AREA AND LIMIT AS SUMS

INTRODUCTION

The process of finding area of some plane region is called quadrature. In this chapter we shall find area bounded by some simple plane curves with the help of definite integral. The question of area becomes simple if we know a rough sketch of the required area.

CURVE TRACTION

In chapter function, we have seen graphs of some simple elementary curves. Here we introduce some essential steps for curve tracing which will enable us to determine the required area.

(i) Symmetry

The curve $f(x, y) = 0$ is symmetrical

* About x-axis if all terms of y contain even powers.
* About y-axis if all terms of x contain even powers.
* About the origin if $f(-x, -y) = f(x, y)$

For example,

$y^2 = 4ax$ is symmetrical about x-axis and $x^2 = 4ay$ is symmetrical about y-axis and the curve $y = x^3$ is symmetrical about the origin.

(ii) Origin

If the equation of the curve contains no constant term then it passes through the origin.

For example $x^2 + y^2 + 2ax = 0$ passes through origin.

(iii) Points of intersection with the axes

If we get real values of x on putting $y = 0$ in the equation of the curve, the real values of x and $y = 0$ give those points where the curve cuts the x-axis. Similarly by putting $x = 0$, we can get the points of intersection of the curve and y-axis. For example, the curve $x^2/a^2 + y^2/b^2 = 1$ intersects the axes at points $(\pm a, 0)$ and $(0, \pm b)$.

(iv) Region

Write the given equation as $y = f(x)$, and find minimum and maximum values of x which determine the region of the curve. For example for the curve $xy^2 = a^2 (a - x)$

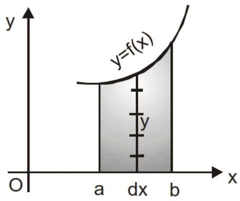
$$y = a\sqrt{\frac{a-x}{x}}$$

Now y is real, if $0 < x \le a$, so its region lies between the lines $x = 0$ and $x = a$

Area bounded by a curve

(A) The area bounded by a cartesian curve $y = f(x)$, x-axis and ordinates $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$$



(B) The area bounded by a cartesian curve $x = f(y)$,

y-axis and abscissa $y = c$ and $y = d$

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

(C) If the equation of a curve is in parametric form, say $x = f(t)$, $y = g(t)$ then the area $= \int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)dt$

SYMMETRICAL AREA

If the curve is symmetrical about a coordinate axis (or a line or origin) then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the desired area.

POSITIVE AND NEGATIVE AREA

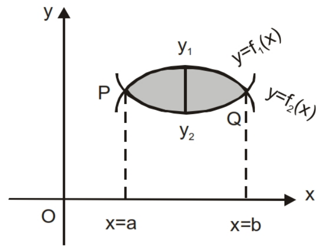
Area is always taken as positive. If some part of the area lies in the positive side i.e., above x-axis and some part lies in the negative side i.e., below x-axis, then the area of two parts should be calculated separately and then add their numerical values to get the desired area.

AREA BETWEEN TWO CURVES

Case I : When two curves intersect at two points and their common area lies between these points.

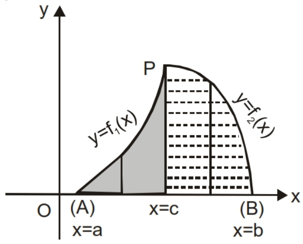
If $y = f_1(x)$ and $y = f_2(x)$ are two curves which intersect at $P(x = a)$ and $Q(x = b)$, and their common area lies between P and Q. then their

$$\text{Common area} = \left|\int_a^b (y_1 - y_2)dx\right| = \left|\int_a^b [f_1(x) - f_2(x)]\,dx\right|$$



Case : II When two curves intersect at a point and the area between them is bounded by x-axis.

If $y = f_1(x)$ and $y = f_2(x)$ are two curves which intersect at P(x=c) and meet x-axis at A(x=a) and B(x=b) respectively, then area between them and x-axis is given by



$$\text{area} = \int_a^c f_1(x)dx + \int_c^b f_2(x)dx$$

LIMIT AS SUMS

For finding sum of an infintie series with the help of definite integration, following formula is used-

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).\frac{1}{n} = \int_0^1 f(x)\,dx$$

The following methode is used to solve the questions on summation of series.

(i) After writing (r – 1)th or rth term of the series, express it in the form $\frac{1}{n} f\left(\frac{r}{n}\right)$. Therefore the given series will take the form $\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{b} \cdot \frac{1}{n}\right)$

(ii) Now writing $\int$ in place of $\left(\lim_{n\to\infty} \sum\right)$, x in place of $\frac{1}{n}$, we get the integral $\int$ f(x) dx in place of above series.

(iii) The lower limit of this integral $= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=0}$

where r = 0 is taken corresponding of first term of the series and uppen limit

$$= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=n-1}$$

where r = n – 1 is taken corresponding to the last term.